Quasi-Inversion of Qubit Channels

2020 International Workshop on Quantum Information,
Quantum Computing and Quantum Control,
Shanghai University, January 2020

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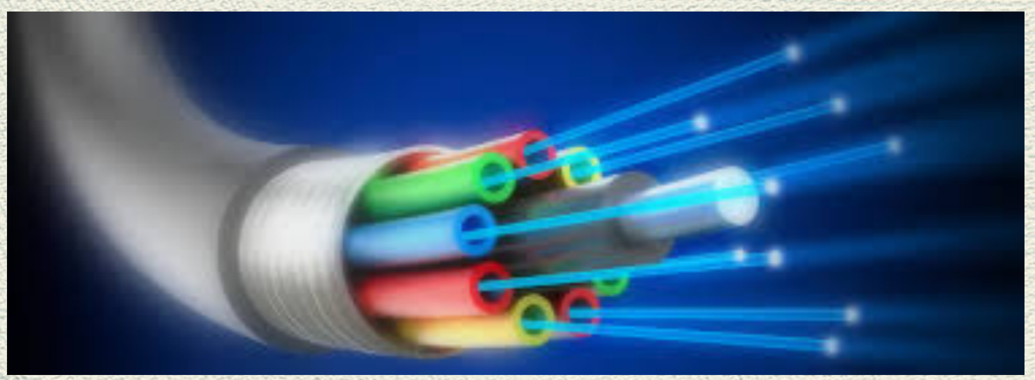
1- Abstract, Motivation and Basic Results

A Quantum Channel:

E

ρ  ρ'

$$E(\rho) = \sum_i K_i \rho K_i^\dagger$$

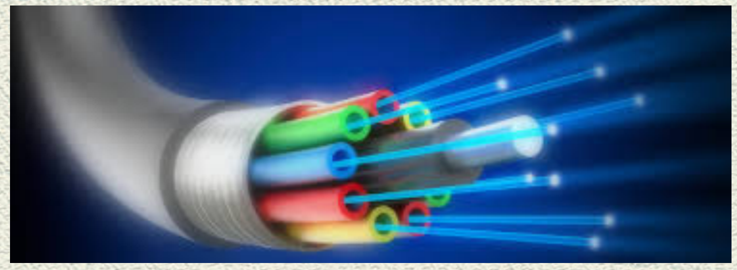
$$\sum_i K_i^\dagger K_i = I$$

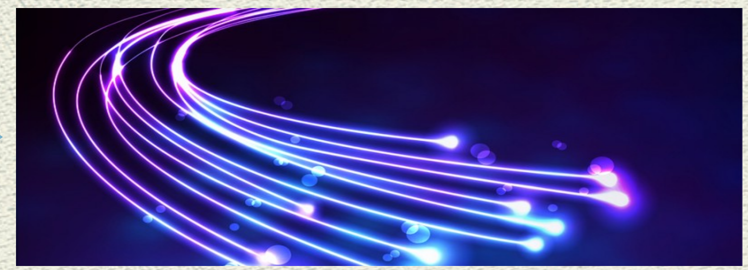
Question:

Can a quantum channel be inverted?

E $\quad$ E^{-1}

ρ



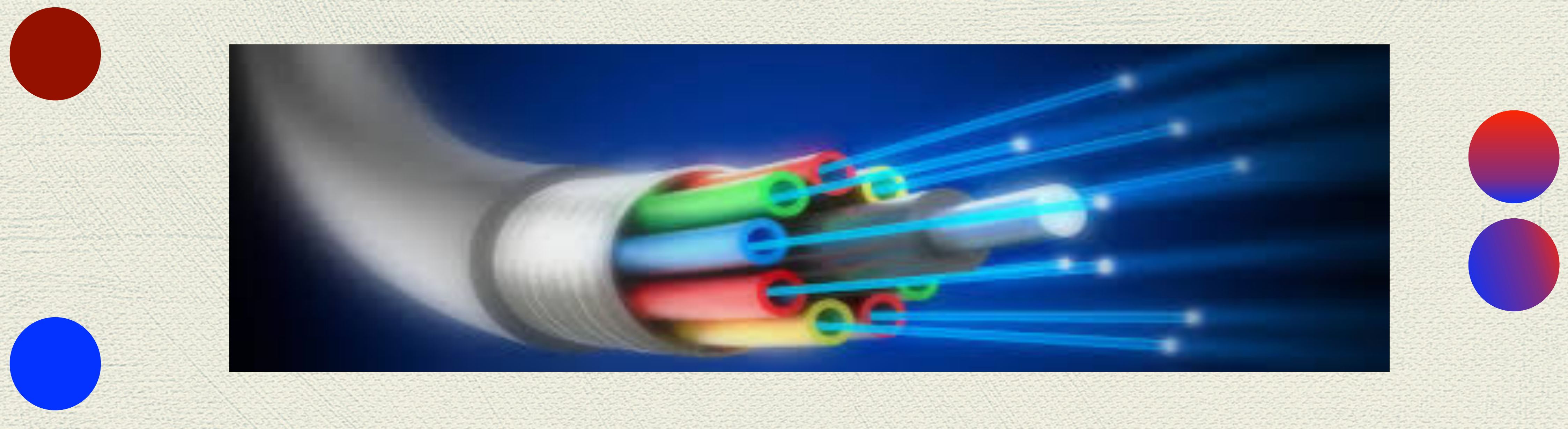


ρ

$$E^{-1}(\rho) = \sum_i L_i \rho L_i^\dagger$$

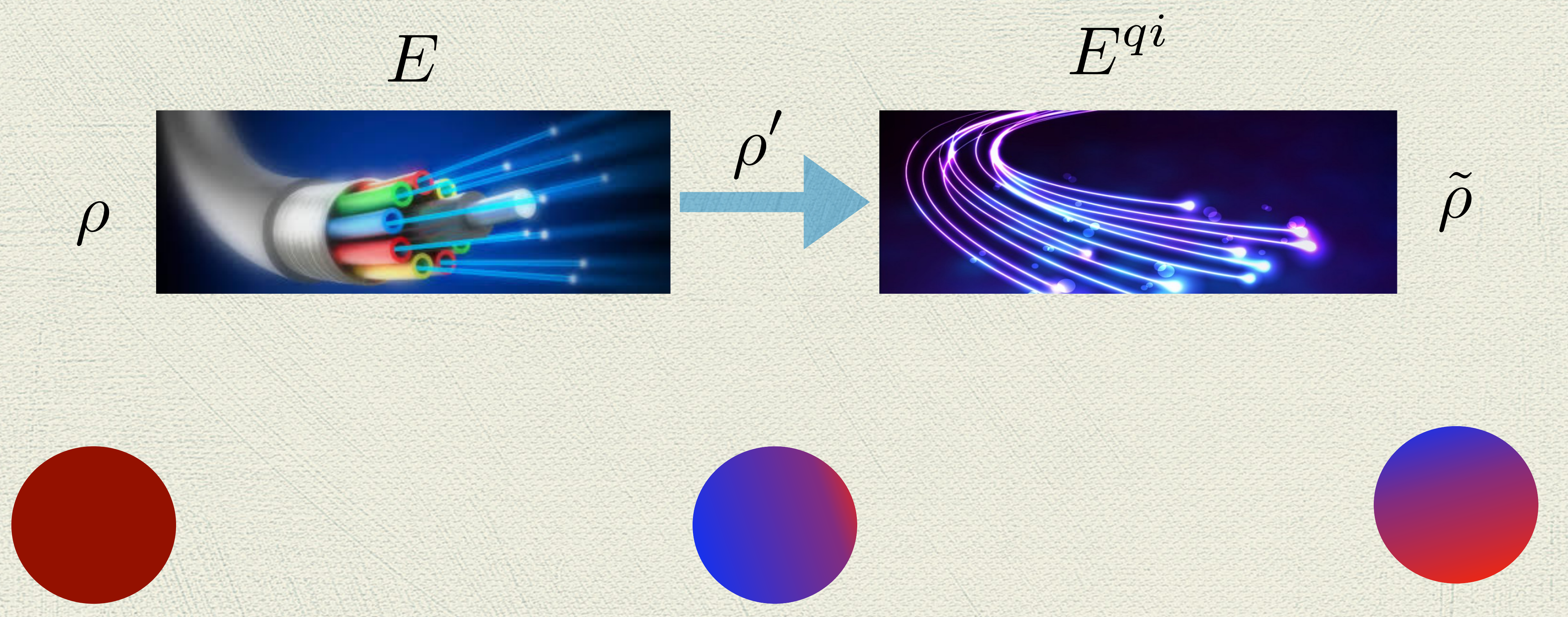
Answer:

It is impossible.

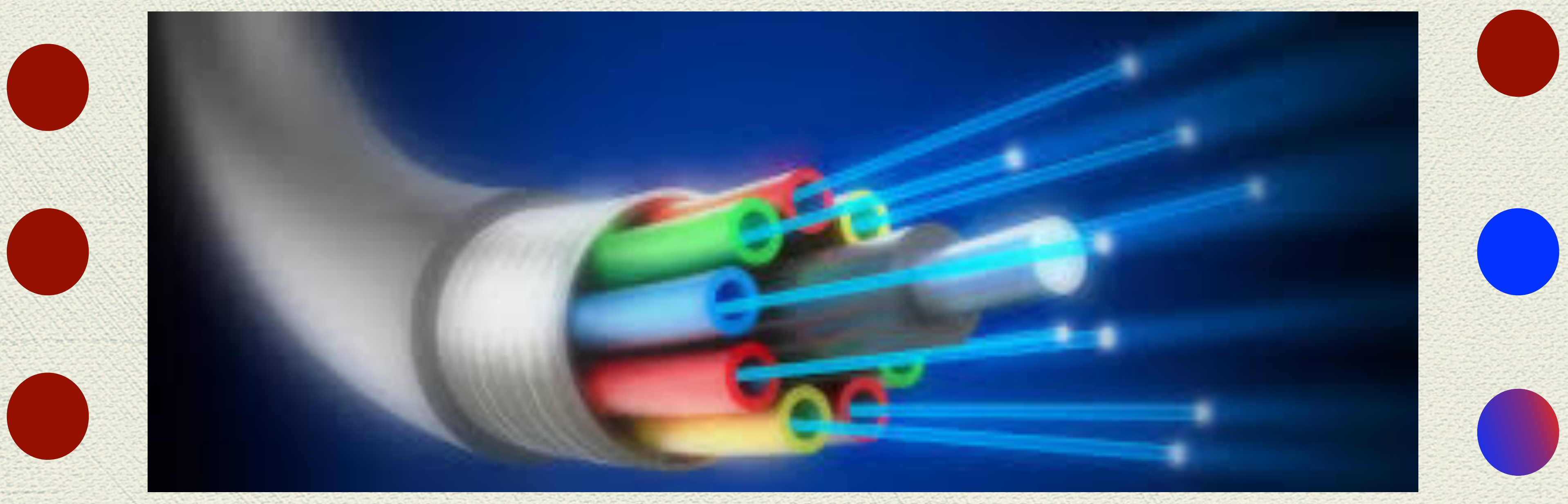


Any channel decreases the distance between quantum states.

Can a quantum channel be quasi-inverted?



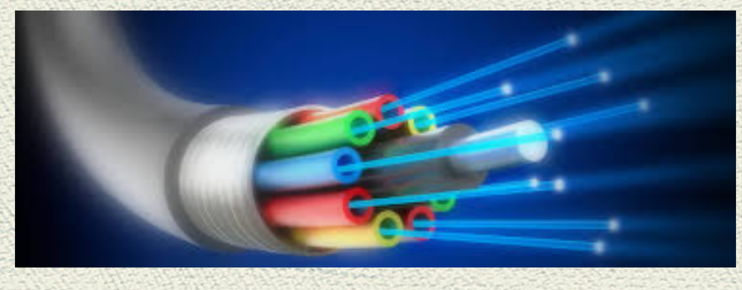
How to measure the performance of a channel?



$$\overline{F}(E) = \int d\phi \langle \phi | E(\rho) | \phi \rangle$$

How to define the Quasi-Inverse

E



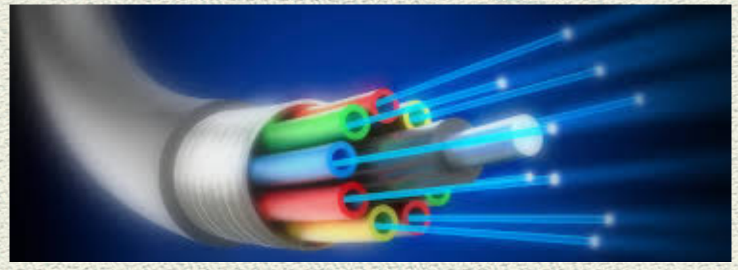
E^{qi}

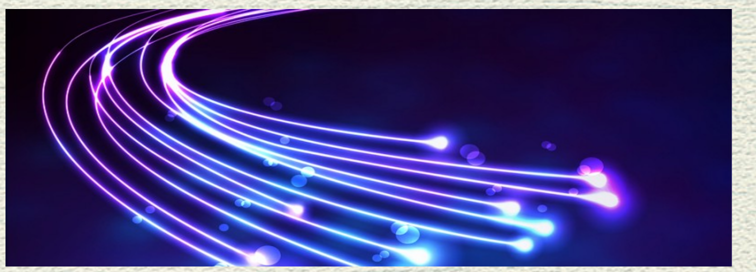
$$\overline{F}(E^{qi} \circ E) \geq \overline{F}(E),$$

$$\overline{F}(E^{qi} \circ E) \geq \overline{F}(\Phi \circ E)$$

Basic Results:

E $\rightarrow$ E^{qi}





1) For qubit channels, the quasi-inverse is always a unitary map

$$\mathcal{E}^{qi}(\rho) = U\rho U^{\dagger}$$

2) It is almost unique.

3) It is both a left and a right quasi-inverse.

4) It can be determined in closed form.

5) Much more difficult for higher dimensional channels.

6) Much much more difficult for classical channels.

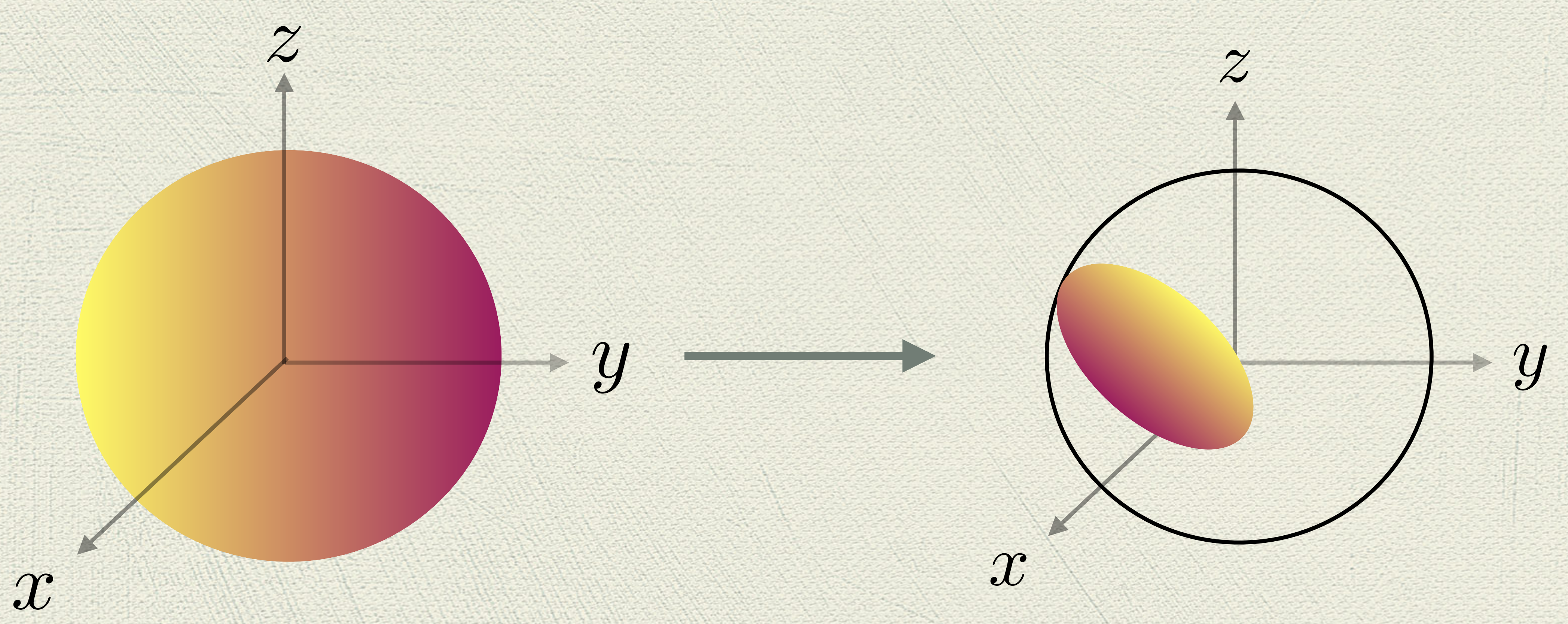
2-Some detail and a few examples

Quasi-inversion of qubit channels:
V. Karimipour, F. Benatti, and R. Floriannini,
arXiv: arXiv:1909.06118

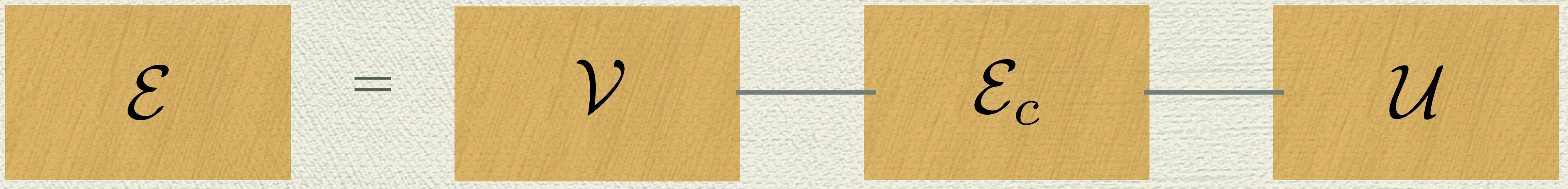
Structure of qubit channels

$$\rho = \frac{1}{2}(I + \mathbf{r} \cdot \sigma) \longrightarrow \rho' = \frac{1}{2}(I + \mathbf{r}' \cdot \sigma)$$

$$\mathbf{r}' = M\mathbf{r} + \mathbf{t}$$

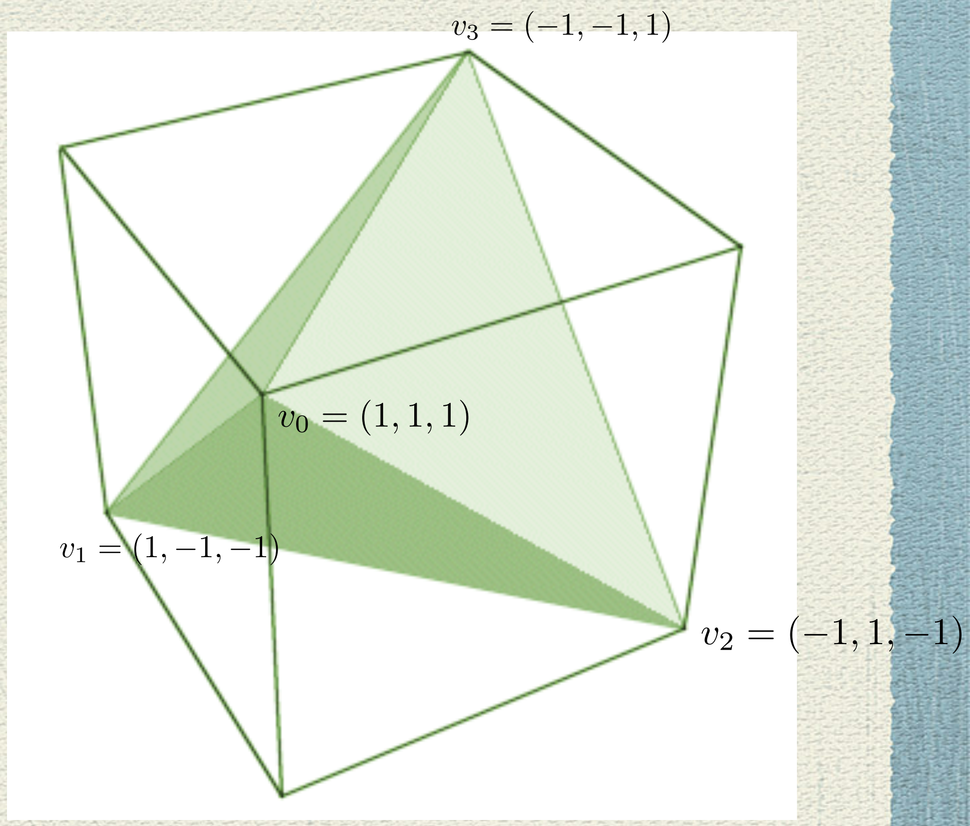


$$\mathcal{E} = \mathcal{U} \circ \mathcal{E}_c \circ \mathcal{V}$$



$$M = O_1 \; \Lambda \; O_2$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

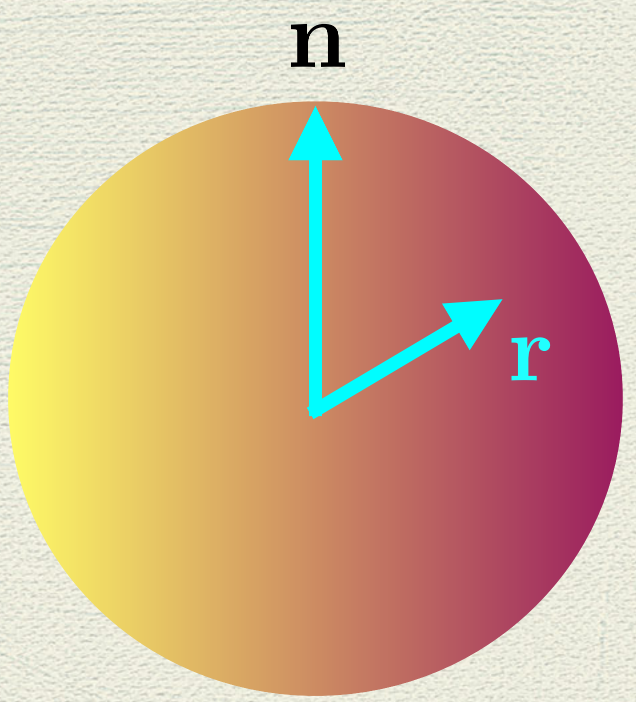


A. Fujiwara, P. Algoet, Phys. Rev. A 59, 3290–3294 (1999).

Mary Beth Ruskai , Linear Algebra and its Applications 347 (2002) 159–187.

Average Fidelity of a Channel

$$|\psi\rangle\langle\psi| = \frac{1}{2}(I + \mathbf{n}\cdot\sigma)$$



$$\mathcal{E}(|\psi\rangle\langle\psi|) = \frac{1}{2}(I + \mathbf{r}\cdot\sigma)$$

$$F = \frac{1}{2}(1 + \mathbf{n}\cdot\mathbf{r})$$

Average Fidelity of a Channel

$$\tfrac{1}{2}(1 + \mathbf{n} \cdot \mathbf{r}) = \frac{1}{2}(1 + \mathbf{n} \cdot M\mathbf{n} + \mathbf{n} \cdot \mathbf{t})$$

$$\int d\mathbf{n}\; n_i = 0$$

$$\int d\mathbf{n}\; n_i n_j = \frac{1}{3}\delta_{ij}$$

$$\overline{F}(E) = \frac{1}{2}(1 + \frac{1}{3}Tr(M))$$

$$(M, t) \rightarrow (N, t')$$

$$=$$

$$(NM, Nt + t')$$

$$\overline{F}(E^{qi} \circ E) = \frac{1}{2}\left(1 + \frac{1}{3}Tr(N_0 M)\right)$$

Theorem: The quasi-inverse of a qubit channel is unitary.

Observation 1:

$$\overline{F}(E) = \int d\phi \langle\phi|E(\rho)|\phi\rangle$$

$$\overline{F}(\sum_i \lambda_i \mathcal{E}_i) = \sum_i \lambda_i \overline{F}(\mathcal{E}_i)$$

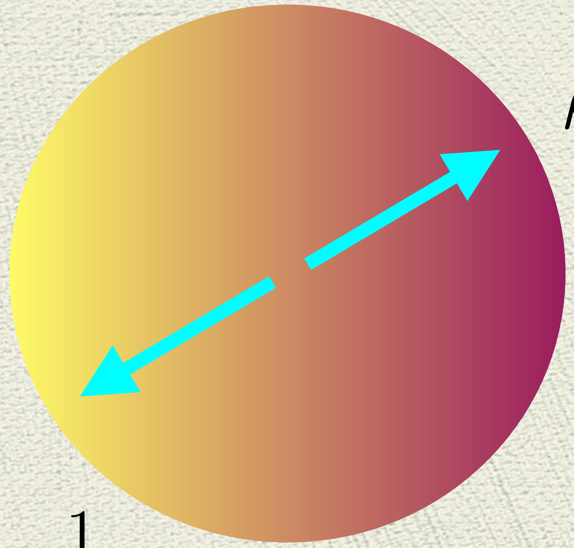
Observation 2: The quasi inverse can be chosen to be unital.

$$E^{qi} = (N_0, t) \qquad E^{qi} = (N_0, 0)$$

(N,t) is a channel → (N,0) is also a channel

This is only true for quit channels.

$$\rho = \frac{1}{2}(1 + \mathbf{r} \cdot \sigma)$$



$$\rho = \frac{1}{2}(1 - \mathbf{r} \cdot \sigma)$$

$$\rho = \frac{1}{3}(I + r\Gamma_z)$$

$r = -1$

$$\rho = \frac{1}{3}\left[1 - \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\right]$$

$r = +1$

$$\rho = \frac{1}{3}\left[1 + \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\right]$$

$r = 1 - \frac{1}{2}$

$$\rho = \frac{1}{3}\left[1 + \frac{1}{2}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\right]$$

Observation 3: Only for qubits, any unital channel is a random unitary channel.

$$E^{qi}(\rho) = \sum_i P_i \; U_i \rho U_i^\dagger$$

$$E^{qi} = \sum_i P_i \; \mathcal{U}_i$$

K. Audenaert, and S. Scheel, On Random Unitary Channels, New J. Phys. 10, 023011 (2008).

It is not true in higher dimensions.

$$\mathcal{E}(\rho) = \frac{1}{j(j+1)}\big(J_x \rho J_x + J_y \rho J_y + J_z \rho J_z\big)$$

L. J. Landau and R. F. Streater, Lin. Algebra Appl. 193, 107 (1993)

The quasi-inverse of a qubit channel is a unitary channel.

$$\overline{F}(\sum_i p_i\, \mathcal{U}_i \circ \mathcal{E}) \geq \overline{F}(\mathcal{E})$$

$$\overline{F}(\mathcal{U}_{max} \circ \mathcal{E}) \geq \overline{F}(\mathcal{E})$$

Example 1: The Pauli Channel

$$E(\rho) = p_0 + p_1\sigma_x\rho\sigma_x + p_2\sigma_y\rho\sigma_y + p_3\sigma_z\rho\sigma_z$$

$$\overline{F}(\mathcal{E}) = \frac{1}{3}(1 + 2p_0)$$

$$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}) = \frac{1}{3}(1 + 2p_{max})$$

$$U = \sigma_{max}$$

U is one of the Kraus operators.

Example 2: The Amplitude Damping Channel

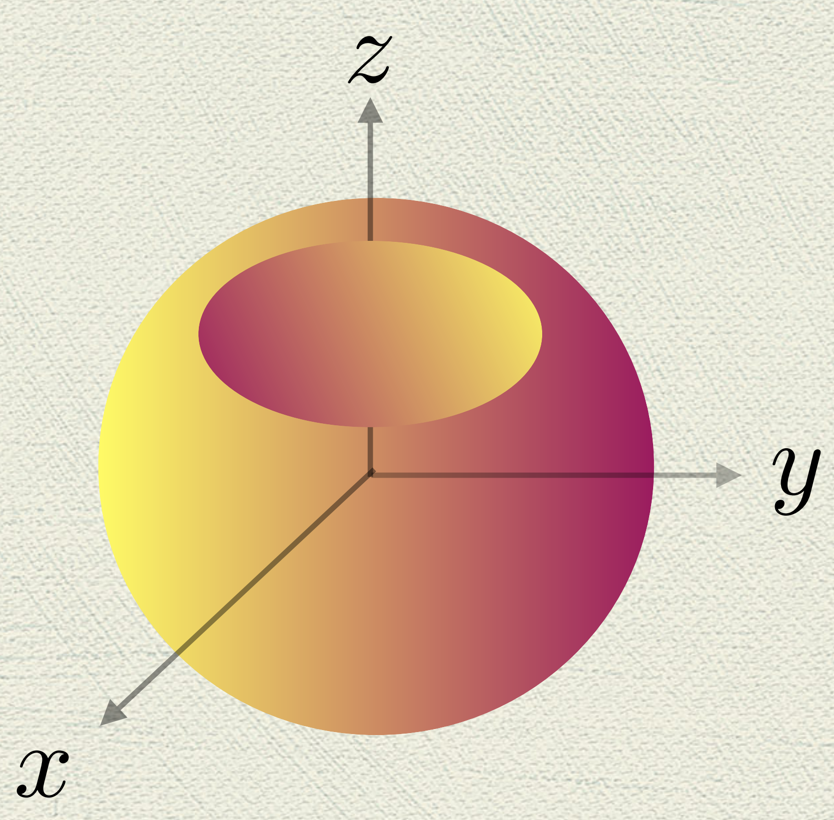
$$E(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & \sqrt{1-\gamma^2} \\ 0 & 0 \end{pmatrix}$$

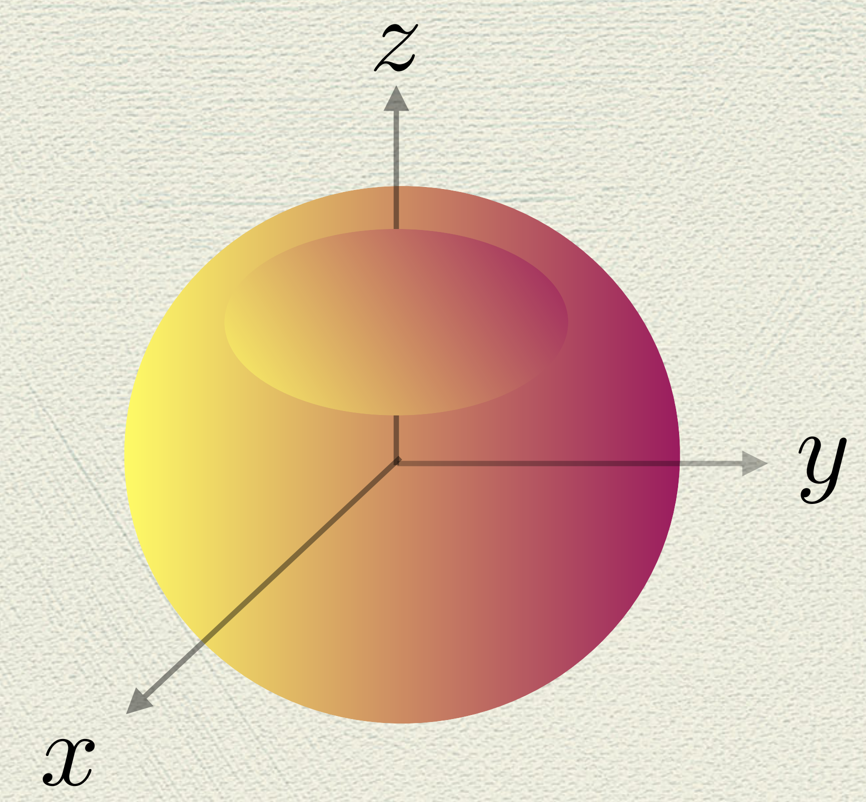
$$\overline{F}(\mathcal{E}_{AD}) = \frac{1}{2} + \frac{1}{6}\gamma^2 + \frac{1}{3}\gamma$$

$$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}_{AD}) = \frac{1}{2} + \frac{1}{6}\gamma^2 - \frac{1}{3}\gamma$$

$$U = \sigma_z$$



$\overline{F}(\mathcal{E}_{AD})$



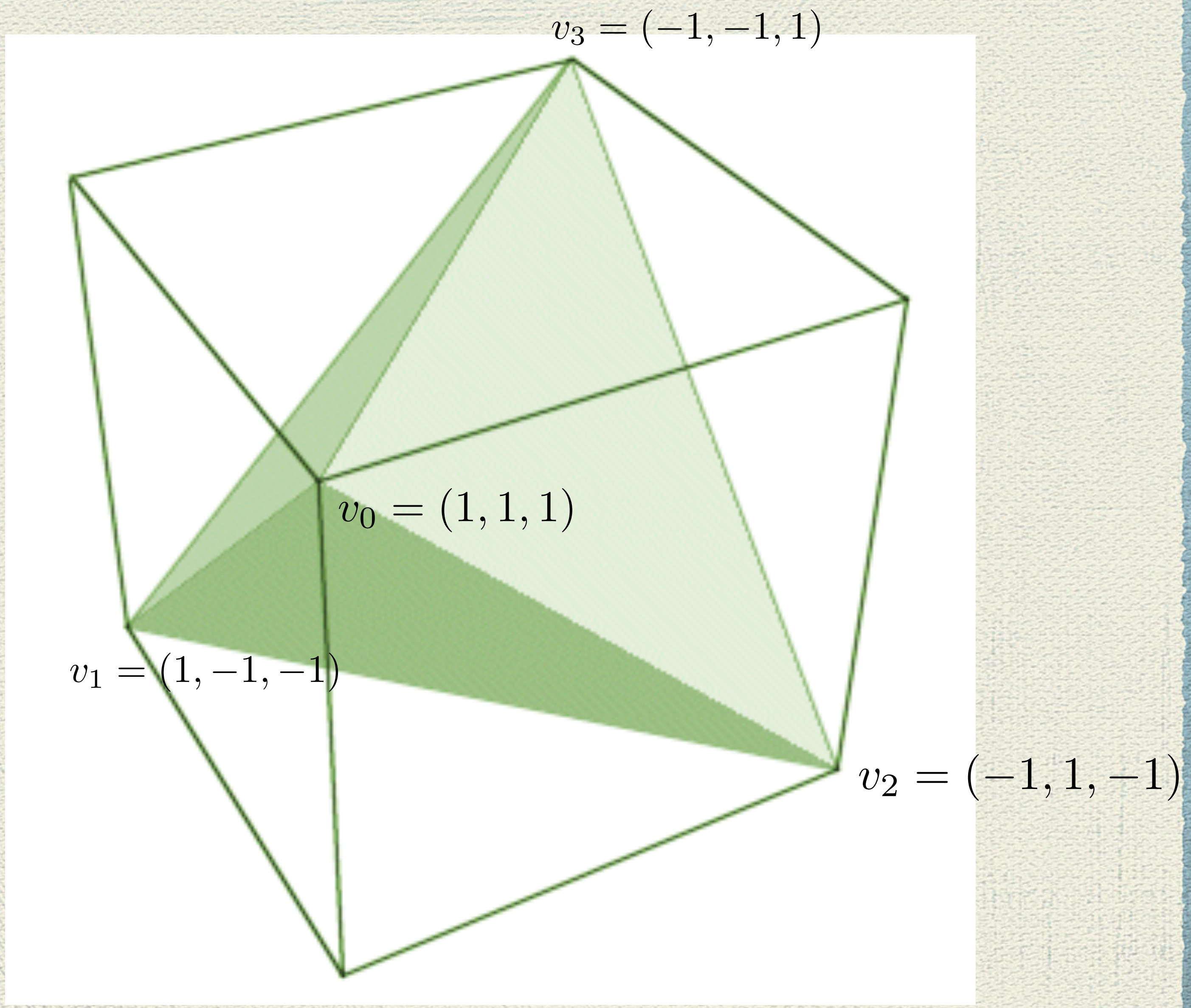
$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}_{AD})$

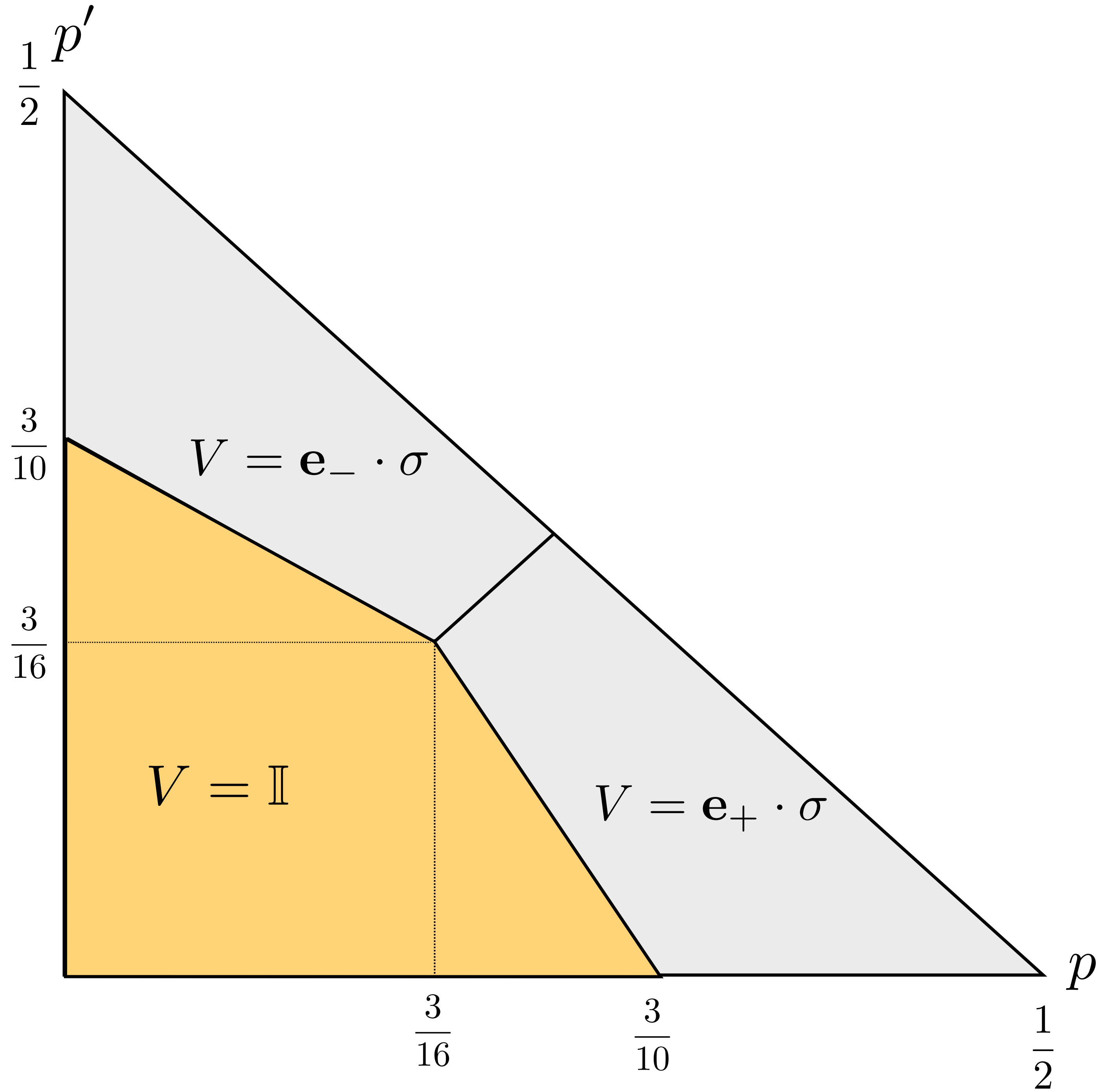
Example 3: The Tetrahedron Channel

$$\mathcal{E}(\rho) = q_\rho + \sum_{\alpha=0}^{3} P_\alpha \, \sigma_\alpha \, \rho \, \sigma_\alpha$$

$$p_0 = p_3 = p$$

$$p_1 = p_2 = p'$$





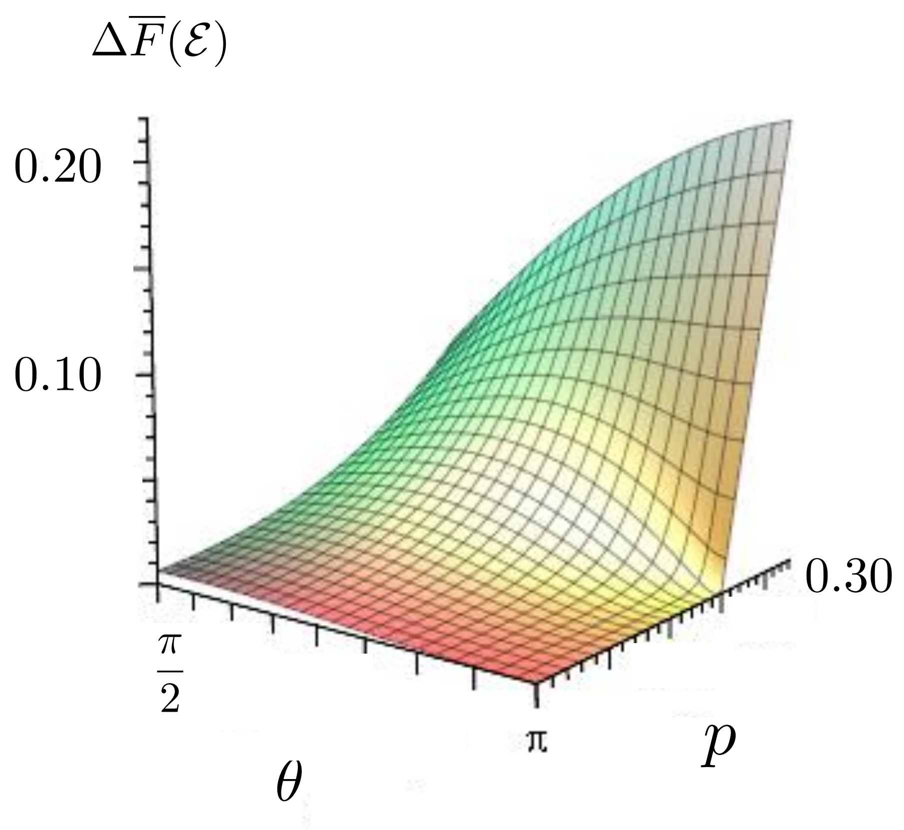
p'
$\frac{1}{2}$
$\frac{3}{10}$
$\frac{3}{16}$
$V = \mathbf{e}_- \cdot \sigma$
$V = \mathbb{I}$
$V = \mathbf{e}_+ \cdot \sigma$
$\frac{3}{16}$
$\frac{3}{10}$
$\frac{1}{2}$
p

Example 4: A Mixed Unitary Channel

$$\mathcal{E}(\rho) = p_0\rho + p\sum_{n=1}^{3} U_n\rho U_n^\dagger$$

$$U_n = e^{-i\frac{\theta}{2}\sigma_n}$$

$$V = e^{i\phi\mathbf{n}\cdot\sigma}$$



A comment on the higher dimensional channels

Any linear Function on a convex set,
takes its maximum values on
the extreme points of the set.

Very little is known about the extreme points of
the space of higher dimensional channels.

The talk of You-Tung Poon: Hybrid quantum error correction

A comment on Classical channels

$$\Omega = \begin{pmatrix} 1-x & y \\ x & 1-y \end{pmatrix}$$

Stochastic Matrix

$$\Omega = \begin{pmatrix} 1-x & x \\ x & 1-x \end{pmatrix}$$

Bi-stochastic Matrix

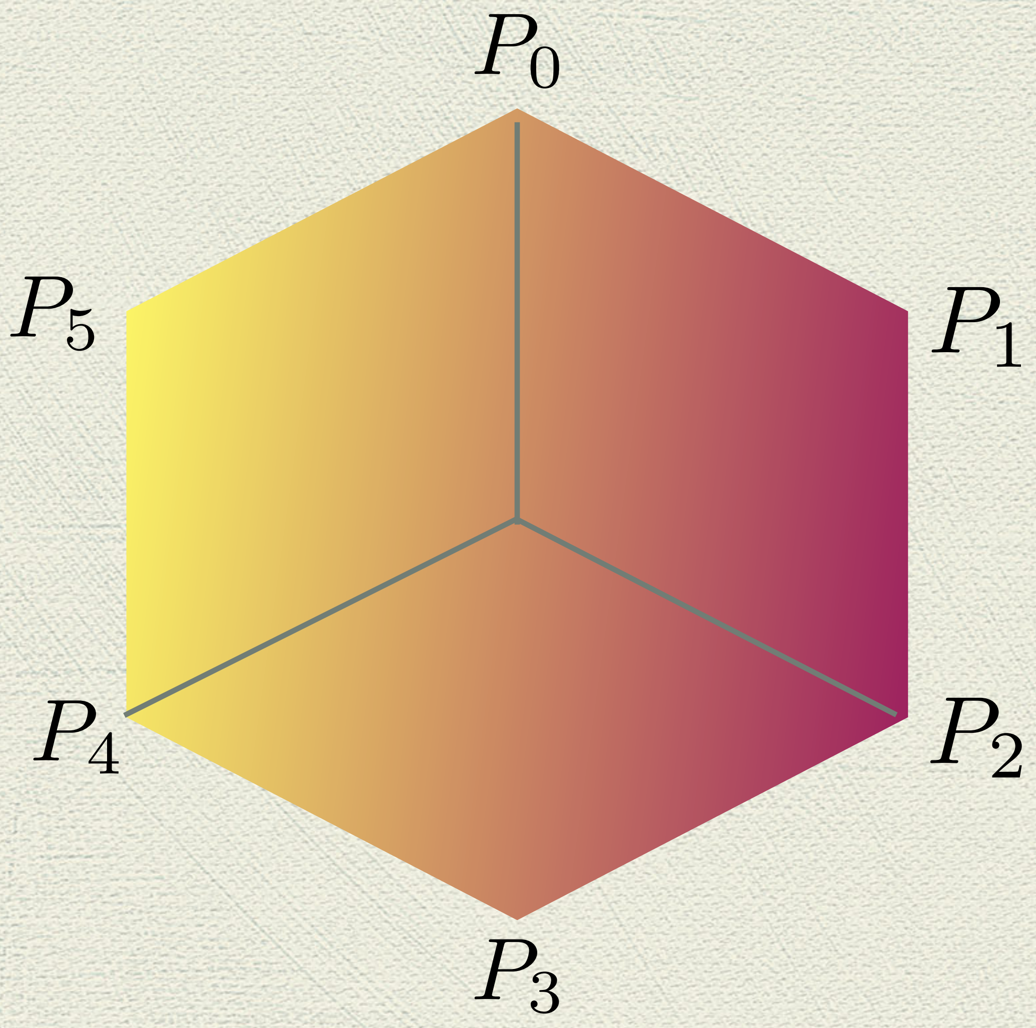
$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Omega = xP_0 + (1-x)P_1 = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}$$

The inverse of a stochastic matrix!

$$\Omega = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix} \qquad \Omega^{-1} = \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}$$

The space of bi-stochastic maps has a simple structure.



Birkhoff Polytope

But the fidelity is a non-linear function

$$F(P,Q) = \sum_i \sqrt{P_i Q_i}$$

Thank you for your attention

The explicit form of the quasi-inverse

$$\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger$$

$$K_i = a_i + \mathbf{b}_i \cdot \sigma$$

$$\overline{F} = \sum_i (a_i^* a_i + \tfrac{1}{3} \mathbf{b}_i \cdot \mathbf{b}_i{}^*)$$

$$\overline{F} = \langle a a^* \rangle + \tfrac{1}{3} \langle \mathbf{b} \cdot \mathbf{b}^* \rangle$$

The explicit form of the quasi-inverse

$$\overline{F} = \langle a a^* \rangle + \tfrac{1}{3}\langle \mathbf{b} \cdot \mathbf{b}^* \rangle$$

Trace-Preserving Property → $\langle a^* a \rangle + \langle \mathbf{b} \cdot \mathbf{b}^* \rangle = 1$

$$\overline{F}(\mathcal{E}) = 1 - \tfrac{2}{3}\langle \mathbf{b} \cdot \mathbf{b}^* \rangle$$

$$B_{\alpha,\beta} = \tfrac{1}{2}\langle b_\alpha b_\beta^* + b_\alpha^* b_\beta \rangle$$

$$\overline{F}(\mathcal{E}) = 1 - \tfrac{2}{3} Tr(B)$$

$$\mathcal{E}^{qi} \circ \mathcal{E} = \sum_i VK_i\rho(VK_i)^\dagger$$

The quasi-inverse $\quad V = x_0 + i\mathbf{x} \cdot \sigma$

$$K_i = a_i + \mathbf{b}_i \cdot \sigma$$

$$\Delta\overline{F}(\mathcal{E}) = \frac{2}{3}\begin{pmatrix} x_0 & \mathbf{x}^t \end{pmatrix} Q \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix}$$

A special subclass

$$\mathcal{E} = \mathcal{V} — \mathcal{E}_c — \mathcal{V}^{-1}$$

Twisting   Un-twisting

$$M = O_1 \, \Lambda \, O_1^t$$

Symmetric affine matrix $M = M^t$

$$Q = \tfrac{1}{2} \begin{pmatrix} 0 & \mathbf{v}^t \\ \mathbf{v} & 2\widehat{B} \end{pmatrix}$$

$$\mathbf{v} = i\langle a^* \mathbf{b} - a \mathbf{b}^* \rangle$$

$$\Delta \overline{F} = \tfrac{2}{3} \ \mathrm{Max} \ (\lambda_{max}, 0)$$